Vv256 Recitation Class

Wang Qian

UM-SJTU Joint Institute

2013-2014 Fall

Personal Info.

Wang Qian

- Office hour: Friday 10 a.m. to 11:40 a.m. at e-reading room
- Recitation class: Wednesday 6 p.m. to 8 p.m. at D300
- e-mail: wangqian1992511@sjtu.edu.cn
- Tel: 18817519155
- Skype: wangqian1992511@gmail.com

When sending an e-mail to me, please begin your subject with "[VV256]", so that I can quickly recognize it. It can also ensure that your mail cannot be filtered, especially for those who use the Tencent Mail. I may not be able to reply your short message at once, but if you have something urgent, please feel free to call me.

Personal Info.

In my recitation class, I will discuss several topics that Professor Liu suggested us to cover during the recitation class. For each topic, I will not only illustrate the concept, but also give you one or two examples to emphasize the importance of calculation. For every assignment, I will briefly talk about it after the due date (if time permitted).

Contents I

Contents II

Contents III

Contents IV

Linear and Nonlinear Equations

A linear ODE should have the form:

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + ... + a_{n-1} \frac{dy}{dt} + a_n y$$

It is called linear because if we regard $\frac{d^n y}{dt^n}$, $\frac{d^{n-1} y}{dt^{n-1}}$, ..., y as variables, then we got a linear function of these variables. In the textbook, there is a comparison between a linear equation and a nonlinear equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \ \frac{d^2\theta}{dt^2} + \frac{g}{L} sin(\theta) = 0$$

The later one is nonlinear because of the appearance of $sin(\theta)$.

Linear and Nonlinear Equations

Are the following equations linear or nonlinear?

(1) $t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+2y=sin(t)$

(2) $(1+y^2)\frac{d^2y}{dt^2}+t\frac{dy}{dt}+y=e^t$

(3) $\frac{d^4y}{dt^4}+\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=1$

(4) $\frac{dy}{dt}+ty^2=0$

(5) $\frac{d^2y}{dt^2}+sin(t+y)=sin(t)$

(6) $\frac{d^3y}{dt^3}+t\frac{dy}{dt}+(cos^2(t))y=t^3$

Linear and Nonlinear Equations

(1) Linear. The term $sin(t)$ does not affect the linearity.
(2) Nonlinear. There is a term $y^2\frac{d^2y}{dt^2}$.
(3) Linear.
(4) Nonlinear. There is a term $ty2$.
(5) Nonlinear. There is a term $sin(t+y)$.
(6) Linear. The term $(cos^2(t))y$ does not affect the linearity.

Linear and Nonlinear Equations

There is a set of theorems about existence and uniqueness of solutions for linear and nonlinear equations. Here, let's just focus on the nonlinear one.

Theorem

Let the functions f *and* $\partial f/\partial y$ *be continuous in some rectangle* $\alpha < t < \beta$, $\gamma < y < \delta$ *containing the point* (t_0, y_0). *Then, in some interval* $t_0 - h < t < t_0 + h$ *contained in* $\alpha < t < \beta$, *there is a unique solution* $y = y(t)$ *of the initial value problem:*

$$\frac{dy}{dt} = f(t, y) \text{ for } y(t_0) = y_0$$

Separable Equations

A separable equation might be nonlinear, and it often leads to implicit solution instead of explicit solution, hence we need to worry about interval of validity. Let's first consider the example during the class with its initial condition changed.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},\ y(0) = 1$$

After we solve the differential equation, we can find the implicit solution:

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

From the initial condition, we can find that $C = -1$. Therefore, the explicit solution is:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}$$

Separable Equations

Notice that now we obtain two solutions satisfying the initial condition. The reason is that the initial point now lies on the line $y = 1$, so no rectangles can be drawn about it within which f and $\partial f / \partial y$ are continuous.
Now let's do an one more example:

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}},\ y(0) = -1$$

We can separate and then integrate both sides to get:

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + C$$

From the initial condition, we can find that $C = -\frac{3}{2}$.

Separable Equations

From this, we can find the implicit solution easily:

$$y^{-2} = 3 - 2\sqrt{1+x^2}$$

Further, we can obtain the explicit solution as:

$$y(x) = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition, we observe that "-" is correct. To get the interval of validity:

$$3 - 2\sqrt{1+x^2} > 0 \Rightarrow -\frac{5}{2} < x < \frac{5}{2}$$

This interval includes the initial condition as well.

Integrating Factor

To solve the general first order linear equation, we should find the integrating factor for it. (Integrating factor will also be used for exact differential equations, which will be covered in the future.)

$$\frac{dy}{dt} + p(t)y = g(t)$$

To determine the integrating factor $\mu(t)$, we multiply the equation above by this as yet undetermined function $\mu(t)$ to obtain:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

Observe that the left hand side will become the derivative of $\mu(t)y$, if:

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

Integrating Factor

This is a simple separable equation. Since the constant term can be chosen arbitrarily, we can just choose it as zero to obtain the integrating factor:

$$\mu(t) = exp(\int p(t)dt)$$

Therefore, we can write the original differential equation as:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t) \Rightarrow \mu(t)y = \int \mu(t)g(t)dt + C$$

Consider the following differential equation as an example:

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

Divide each side by (x^2+1) to obtain:

$$\frac{dy}{dx} + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

Integrating Factor

Now, we can directly determine the integrating factor as:

$$\mu(t) = exp(\int \frac{3x}{x^2+1}dx) = (x^2+1)^{\frac{3}{2}}$$

Plug the integrating factor into the original equation:

$$(x^2+1)^{\frac{3}{2}}\frac{dy}{dx} + 3x(x^2+1)^{\frac{1}{2}}y = 6x(x^2+1)^{\frac{1}{2}}$$

Here, we should notice that, as what we want:

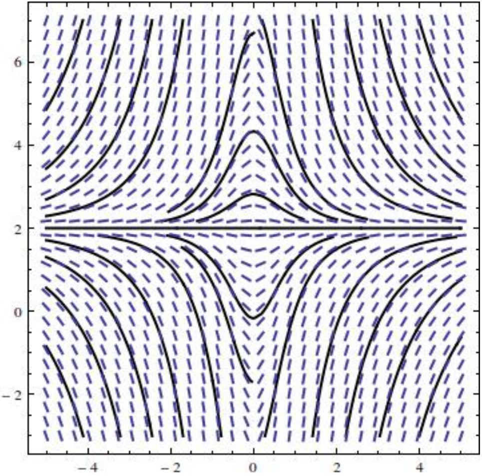
$$\frac{d}{dx}((x^2+1)^{\frac{3}{2}}y) = 6x(x^2+1)^{\frac{1}{2}}$$

Integrate each side, and divide each side by the integrating factor:

$$y(x) = 2 + C(x^2+1)^{-\frac{3}{2}}$$

Integrating Factor

Here shows the direction field and the integral curves:



Modeling with First Order Equations

During the lecture, we discussed an example about compound interest. More generally, this is a problem about natural growth, which can be solved by separating variables:

$$\frac{dx}{dt} = kx,\ x(0) = x_0 \Rightarrow x(t) = x_0 e^{kt}$$

Now, let's have a look at the mixing problem, which let you practice integrating factors. At $t = 0$ a tank with the volume of $V(0) = V_0$ contains salt with the mass of $x(0) = x_0$. Assume that the water containing salt with the concentration of c_i g/L is entering the tank at a rate of r_i L/s. Also, the water in the tank is leaving the tank at the rate of r_o L/s.
For a short period of time Δt:

$$\Delta x = r_i c_i \Delta t - r_o c_o \Delta t$$

Therefore, we derive that $(V(t) = V + (r_i - r_o)t)$:

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V(t)} x \text{ with the integrating factor } \mu(t) = exp(\frac{r_o}{V(t)})$$

Euler's Method I

Suppose we have a first order initial value problem:

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0$$

It is not also easy to solve this problem by using the methods we talked about until now. However, with the help of computer, we can get an approximation value for $y = y(t)$. Today, we only focus on the simplest Euler's Method (without any improvements on it).

Notice that here we require that both f and $\frac{\partial f}{\partial y}$ should be continuous.

Euler's Method I

For a point $t = t_1$ near $t = t_0$, we can determine $y(t_1) = y_1$ by:

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

We can repeat this step for any further points. The following is the pseudocode for Euler's Method:

Algorithm

procedure *approx*($f(t,y)$, t_0, y_0, t_n, y_n)
  {*step is a real number small enough*}
  $t := t_0;\ y := y_0$
  while $t \neq t_n$ **do**
    $y := y + step * f(t,y);$
    $t := t + step;$
  $y_n := y;$

Actually, we can improve this method slightly to obtain a better result. But, we will talk about this in the future.

Euler's Method I

Let's still use the example:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},\ y(0) = -1 \Rightarrow y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

The result shows us that $y(100) \approx -1009$. With the help of MATLAB, we can find that the computed values become more accurate as the step size decreases:

Exact	step = 1	step = 0.1	step = 0.01	step = 0.001
-1009.0515	-1007.1766	-1010.3683	-1009.0326	-1009.0495

Substitution

By now, we only discussed separable or linear first order ODE. But there are still many differential equations neither separable nor linear. Under this circumstance, substitution may be needed. Let's try one example problem:

$$\frac{dy}{dx} = \frac{1}{xysin^2(xy^2)} - \frac{y}{2x}$$

Here, what bothers us is the term $sin^2(xy^2)$. So, a common method is to substitute xy^2 with u. From $u = xy^2$, we can derive that:

$$\frac{du}{dx} = \frac{d}{dx}xy^2 = y^2 + 2xy\frac{dy}{dx} = \frac{2}{sin^2 u}$$

This is a separable equation, and we can solve it to get:

$$u - \frac{1}{2}sin(2u) - 4x + C = 0$$
$$xy^2 - \frac{1}{2}sin(2xy^2) - 4x + C = 0$$

Substitution

What about this one?

$$\frac{dy}{dx} = \frac{y}{2x} + \frac{1}{2y} tan\frac{y^2}{x}$$

The annoying part becomes $\frac{y^2}{x}$, now. If we substitute it with u, we can obtain:

$$\frac{du}{dx} = \frac{d}{dx}\frac{y^2}{x} = -\frac{y^2}{x^2} + \frac{2y}{x}\frac{dy}{dx} = \frac{tan(u)}{x}$$

Now, we get a separable equation again. Solve to get:

$$sin(u) = Cx$$
$$sin\frac{y^2}{x} = Cx$$

Homogeneous Equations

Homogeneous first order differential equations have a form of:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

This is just a special case of the substitution method. For these kinds of equations, we should substitute $\frac{y}{x}$ with u to get a separable equation with a form of:

$$x\frac{du}{dx} = F(u) - u$$

Let's look at the first example:

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

This equation is neither separable nor linear, so we substitute $\frac{y}{x}$ with v to obtain:

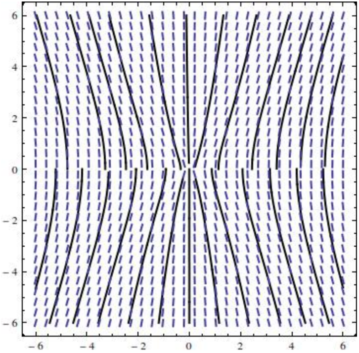
$$v + \frac{dv}{dx} = \frac{2}{v} + \frac{3v}{2}$$

Homogeneous Equations

From this separable equation, we can derive that:

$$y^2 + 4x^2 = kx^3 \Rightarrow y(x) = \pm\sqrt{kx^3 - 4x^2}$$

This means that when $k > 0$, then $x > \frac{4}{k}$ and when $k < 0$, then $x < \frac{4}{k}$.
Here shows the direction field and the integral curves:



Homogeneous Equations

Let's try an IVP.

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2},\ y(x_0) = 0\ (x_0 > 0)$$

Divide both side by x to get a homogeneous equation. Use the similar method to get the solution:

$$sin^{-1}v = ln(x) + C$$

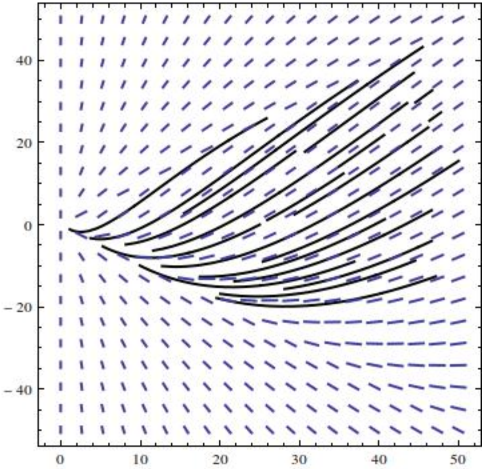
Since $v(x_0) = \frac{y(x_0)}{x_0} = 0$, we have $C = -ln(x_0)$. That is to say:

$$y(x) = sin(ln\frac{x}{x_0})$$

Since $sin^{-1}v = ln(\frac{x}{x_0})$, we can derive that $x_0e^{-\pi/2} \leqslant x \leqslant x_0e^{\pi/2}$. Also, we should notice that the solution should be stricted to the triangle area $x \geqslant \mid y \mid$.

Homogeneous Equations

Here shows the direction field and the integral curves:



Bernoulli Equations

Bernoulli equations have a form of:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \ (n \neq 0, 1)$$

This is also a special case of the substitution method. For these kinds of equations, we should substitute y^{1-n} with u to get a linear equation with a form of:

$$x\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Let's look at the first example:

$$x\frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$$

Divide both side by x to get a Bernoulli Equation and substitute $y^{-\frac{1}{3}}$ with v to obtain:

$$\frac{dv}{dx} - \frac{2}{x}v = -1$$

Bernoulli Equations

From this linear equation, it is possible for us to get the solution:

$$y(x) = \frac{1}{(x + Cx^2)^3}$$

The key point for this substitution is to make both v and $\frac{dv}{dx}$ appearing in the linear equation. This kind of thought is also useful in some equations other than Bernoulli Equations.

$$2xe^{2y}\frac{dy}{dx} = 3x^4 + e^{2y}$$

Here, if we substitute e^{2y} with v, we can find that $\frac{dv}{dx}$ also appears in the equation. Therefore, the equation will get a form of:

$$\frac{dv}{dx} = \frac{v}{x} + 3x^3$$

This linear equation can also be solved.

Exact Equations

If we can always find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$, then $F(x,y) = C$ is a general solution for the equation:

$$M(x,y)dx + N(x,y)dy = 0$$

In this condition, this equation is called as an exact equation. Actually, we should check whether M_y is equal to N_x to judge whether the equation is exact or not. For instance, the first equation is exact, but the second one is not (though they have the same solution):

$$y^3dx + 3xy^2dy = 0$$
$$ydx + 3xdy = 0$$

Exact Equations

We should follow the steps below to solve this kind of problem:
(1) $F(x,y) = \int M(x,y)dx + g(y)$
(2) $g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$
(3) obtain $g(y)$ and plug it into $F(x,y)$
The following problem can be an example to illustrate these steps:

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

We can examine that this equation is exact, since $M_y = N_x = 6x - 3y^2$. From step (1), we integrate $\frac{\partial F}{\partial x} = M$ to get:

$$F(x,y) = \int (6xy - y^3)dx = 3x^2 - xy^3 = g(y)$$

Then, we differentiate this equation with respect to y. From $\frac{\partial F}{\partial y} = N$:

$$\frac{\partial F}{\partial y} = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

Exact Equations

Finally, after we get $g(y)$, we can obtain that:

$$F(x, y) = 3x^2y - xy^3 + 2y^2 + C$$

What can we do if M_y is not equal to N_x? We should use the strategy of integrating factor again. If we multiply the equation by μ, then we will get an exact equation:

$$\mu M(x, y)dx + \mu N(x, y)dy = 0$$

Let's first suppose that μ **only depends on** x, so:

$$(\mu M)_y = (\mu N)_x \Rightarrow \mu M_y = \mu N_x + N\frac{d\mu}{dx} \Rightarrow \mu(x) = exp(\int \frac{M_y - N_x}{N}dx)$$

Similarly, if μ **only depends on** y:

$$\mu(y) = exp(\int \frac{N_x - M_y}{M}dy)$$

Exact Equations

Also, we can find an integrating factor when μ **only depends on** $\phi(x, y)$:

$$\mu(x, y) = exp(\int \frac{N_x - M_y}{M\phi_y - N\phi_x} d(\phi(x, y))$$

Let's only do an example problem for this condition.

$$(3x + \frac{6}{y})dx + (\frac{x^2}{y} + 3\frac{y}{x})dy = 0$$

We can calculate that:

$$\frac{N_x - M_y}{xM - yN} = \frac{1}{xy}$$

Therefore, the integrating factor is:

$$\mu(xy) = exp(\int \frac{1}{xy} d(xy)) = xy$$

Euler's Method II

Last week, we learnt that Euler's Method can be used to obtain a numerical approximation for a first order initial value problem. For a point $t = t_1$ near $t = t_0$, we can determine $y(t_1) = y_1$ by:

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

Here, we just use the leftmost point of a certain interval to approximate the value. Why don't we take an average of the gradients for both ends of this interval. This is the basic thought to improve Euler's Method:

$$y_{tmp} = y_0 + f(t_0, y_0)(t_1 - t_0)$$
$$y_1 = y_0 + [f(t_0, y_0) + f(t_1, y_{tmp})]\frac{(t_1 - t_0)}{2}$$

Euler's Method II

You can implement this method by yourself with the help of MATLAB. If you can compare the result with the example last week, you will find that the improvement is very obvious.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)},\ y(0) = -1 \Rightarrow y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Exact	step = 10	step = 5	step = 1	step = 0.5
-1009.0515	-1018.9803	-1010.0581	-1009.0769	-1009.0577

Second Order Linear Homogeneous Equations

Let's first focus on the second order linear equation with the form of:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0$$

The characteristic equation for this differential equation is:

$$\lambda^2 + a\lambda + b = 0$$

According to the value of the discriminant $\Delta = a^2 - 4b$, we can consider the following three distinct conditions:

$$\begin{array}{ll} \Delta > 0 & y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ \Delta < 0 & y(t) = C_1 e^{at} cos(bt) + C_2 e^{at} sin(bt) \\ \Delta = 0 & y(t) = (C_1 + C_2 t) e^{\lambda t} \end{array}$$

General Second Order Linear Homogeneous Equations

Now, we can discuss a more general case:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

Linear homogeneous equations have many useful properties.

Theorem

If y_1 and y_2 are two solutions of the differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constant c_1 and c_2.

Second Order Linear Homogeneous Equations

The proof for this theorem is easy and obvious.

Proof.

Let $y = c_1y_1 + c_2y_2$, then $y' = c_1y_1' + c_2y_2'$ and $y'' = c_1y_1'' + c_2y_2''$. Hence:

$$\begin{aligned} y'' + py' + q &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= (c_1y_1'' + c_2y_2'') + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

That is to say, $y = c_1y_1 + c_2y_2$ is a solution of the differential equation.

Second Order Linear Homogeneous Equations

For example, from observation, we can find that $y = cos(t)$ and $y = sin(t)$ are two solutions for the differential equation $y'' + y = 0$. Then the general solution for this differential equation can be expressed as $y = c_1 cos(t) + y_2 sin(t)$.
Furthermore, there is also a theorem about existence and uniqueness for the second order homogeneous (and even nonhomogeneous) linear equation. The proof is not required, but the conclusion is important.

Theorem

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),\ y(t_0) = y_0,\ y'(t_0) = y'_0$$

where p, q and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = y(t)$ of this problem, and the solution exists throughout the interval I.

Second Order Linear Homogeneous Equations

The importance of the valid interval should be emphasized again, since it is always ignored when solving a problem. Let's find the longest interval where a unique solution of initial value problem exists.

$$(1-t^2)y'' + ty' = ln(t),\ y(\frac{1}{2}) = 1,\ y'(\frac{1}{2}) = 0$$

According to the form of the differential equation in theorem of existence and uniqueness, we find that:

$$p(t) = \frac{t}{1-t^2},\ q(t) = 0,\ h(t) = \frac{ln(t)}{1-t^2}$$

So the common interval for p, q and h to be continuous is (0,1) or (1,∞). Consider the initial value condition, we should choose the interval of (0,1). When you are solving other problems, it is also necessary to think what is the valid interval.

Second Order Linear Homogeneous Equations

Now, consider the equation $y'' + y = 0$ discussed in the previous slide. If we have the initial condition $y(0) = 3$ and $y'(0) = 2$, we will obtain that the unique solution for this problem is $y(t) = 3cos(t) + 2sin(t)$.

From the procedure above, we find that what we should do at the beginning is always to find such two y_1 and y_2 with "different essence". Actually, it is required that y_1 and y_2 should be linearly independent. That is to say, there exist no constants, such that $y_1 = ky_2$.

This concept has already been covered in VV255. Notice that zero is linearly dependent with any other functions.

Wronskian

We now know that we should find two linearly independent solutions. In fact, an easy method is just to find

$$y_1(t_0) = 1,\ y_1'(t_0) = 0,\ y_2(t_0) = 0,\ y_2'(t_0) = 1$$

for an initial point t_0. As long as we find y_1 and y_2, we can plug them into any initial condition.

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{aligned}$$

We will find that the value of c_1 and c_2 depends on $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$. This is why we should introduce Wronskian of function f and g:

$$W(f,g) = \begin{vmatrix} \text{f} & \text{g} \\ \text{f'} & \text{g'} \end{vmatrix} = fg' - f'g$$

Wronskian

The discussion in the previous slide shows that:

Theorem

If y_1 and y_2 are two solutions of the differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

and the Wronskian $W(y_1, y_2)$ is not zero at the point t_0 where the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

are assigned. Then there is a choice of the constants c_1 and c_2 for which $y = c_1y_1 + c_2y_2$ satisfies this differential equation and the corresponding initial condition.

Wronskian

There is also a question that are all solutions of this homogeneous equation in the family of $y = c_1 y_1 + c_2 y_2$. The short answer is yes.

Theorem

If y_1 and y_2 are two solutions of the differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

on the interval I where p and q are continuous and the Wronskian $W(y_1, y_2)$ is not zero for some point t_0 on this interval. Then the family of solution $y = c_1 y_1 + c_2 y_2$ includes every solution of this homogeneous equation.

Maybe it is not so straightforward. So, let's prove this theorem.

Wronskian

Proof.

Let $\psi(t)$ be any solution of this homogeneous equation, and $y_0 = \psi(t_0)$, $y_0' = \psi(t_0)$. Since $W(y_1, y_2)(t_0) \neq 0$, there exist c_1 and c_2 such that $y = c_1 y_1 + c_2 y_2$ is a solution of this initial value problem. Because p and q are continuous on (a, b), the existence and uniqueness theorem guarantees that this solution is unique. Hence, $\psi(t) = c_1 y_1(t) + c_2 y_2(t)$ holds for all $t \in (a, b)$.

Since ψ is arbitrary, every solution can be represented as one form of $c_1 y_1(t) + c_2 y_2(t)$ according to the initial value condition.

Wronskian

The following is a significant theorem about Wronskian.

Theorem

If f and g are differentiable functions on an open interval I, and if $W(f,g)(t_0) \neq 0$ for some point t_0 on I, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f,g) = 0$ for every t on I.

Notice that all these two implications are in one direction only. Let's start from the easier second condition.

Proof.

This condition is easy to prove. Let $f = kg$, so that f and g are linearly dependent. Then, we will get that $W(f,g) = kgg' - kg'g = 0$ for every t on I.

Wronskian

The first condition is not so easy to prove. But it is still not too difficult to understand.

Proof.

Consider the linear combination and suppose that this expression is zero throughout the interval. So, at the point t_0, we have:

$$k_1 f(t_0) + k_2 g(t_0) = 0$$
$$k_1 f'(t_0) + k_2 g'(t_0) = 0$$

Since $W(f,g) \neq 0$ by hypothesis, we can derive that $k_1 = k_2 = 0$. That is to say, f and g are linearly independent.

Abel's Theorem

There is an amazing theorem that gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem

If y_1 and y_2 are two solutions of the differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

on the interval I where p and q are continuous, then the Wronskian $W(y_1, y_2)$ is given by

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)exp(-\int_{t_0}^{t} p(t)dt)$$

Further, $W(y_1, y_2)$ is either zero or never zero for all $t \in I$.

Abel's Theorem

Proof.

We rearrange the following two equations:

$$y_1'' + py_1' + qy_1 = 0$$
$$y_2'' + py_2' + qy_2 = 0$$

to obtain the following equations:

$$(y_1y_2'' - y_1''y_2) + p(y_1y_2' - y_1'y_2) = 0$$
$$W' + pW = 0$$

This is a simple first order linear equation. Solve to get:

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)exp(-\int_{t_0}^{t} p(t)dt)$$

Also, if $W(y_1, y_2)(t_0)$ is not equal to zero, then $W(y_1, y_2)(t)$ will not be equal to zero for all $t \in I$.

Abel's Theorem

Given one solution of the second order linear homogeneous equation find its fundamental set of solution.

$$y'' + \frac{t}{1+t^2}y' - \frac{1}{1+t^2}y = 0,\ y_1 = t$$

To solve this problem, the Abel's Theorem is required. Since $y_1(0) = 0$ and $y_1'(0) = 1$, let y_2 be a solution such that $y_2(0) = 1$ and $y_2'(0) = -1$. By the existence and uniqueness theorem, there is only one such solution y_2 and it is defined on $\mathbb{R}$. By the Abel's Theorem:

$$W(y_1, y_2)(t) = W(y_1, y_2)(0)exp(-\int_0^t \frac{t}{1+t^2}dt) = -\frac{1}{\sqrt{1+t^2}}$$

Hence, y_2 satisfies the first order differential equation

$$ty_2' - y_2 = -\frac{1}{\sqrt{1+t^2}}$$

With the initial condition $y_2(0) = 1$, we obtain $y_2 = \sqrt{1+t^2}$. That is to say, the fundamental set of solution should be $\{t, \sqrt{1+t^2}\}$.

Abel's Theorem

By now, we notice the following equivalent statements:

Theorem

Let y_1 and y_2 be the solution of $y'' + py' + q = 0$ on an open interval I, where p and q are continuous, then
(1) y_1 and y_2 form the fundamental set of solution of this homogeneous equation on I
(2) y_1 and y_2 are linearly independent
(3) $W(y_1, y_2)(t_0) \neq 0$ at some $t_0 \in I$
(4) $W(y_1, y_2)(t) \neq 0$ for all $t \in I$

Reduction of Order

Suppose that we know one nontrivial solution y_1 to the differential equation:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

Then, we can use the Abel's Theorem to get the second solution from the previous slides. Sometimes, the method of reduction of order may also seem to be useful to solve this kind of problem. Let the second solution be $y_2 = v \cdot y_1$, then:

$$y_2' = v' \cdot y_1 + v \cdot y_1'$$
$$y_2'' = v'' \cdot y_1 + 2v' \cdot y_1' + v \cdot y_1''$$

Substitute y, y' and y'' in the original differential equation and collect terms to obtain:

$$y \cdot v'' + (2y_1' + p \cdot y_1)v' + (y_1'' + p \cdot y_1' + q \cdot y_1)v = 0$$

Reduction of Order

Notice that the term $(y_1'' + p \cdot y_1' + q \cdot y_1)v$ can be ignored:

$$y \cdot v'' + (2y_1' + p \cdot y_1)v' = 0$$

By now, suppose that $u = v'$ we will get a first order differential equation:

$$y \cdot u' + (2y_1' + p \cdot y_1)u = 0$$

Therefore, we can solve this equation for u and get v by one more integration. One valid solution for y_2 can be:

$$y_2 = y_1 \int \frac{exp(-\int pdt)}{y_1^2} dt$$

Let's follow an example.

Reduction of Order

Given a differential equation

$$2t^2y'' + 3ty' - y = 0,\ y_1 = t^{-1}$$

Notice that the $p(t)$ means $\frac{3}{2t}$. Therefore:

$$y_2 = y_1 \int \frac{exp(-\int \frac{3}{2t} dt)}{y_1^2} dt = t^{-1} C \cdot \int t^{\frac{1}{2}} dt = C \cdot t^{\frac{1}{2}}$$

Second Order Linear Nonhomogeneous Equations

By now, we only talk about solving the homogeneous equations. Before we introduce the method to solve nonhomogeneous equations, let's first focus on the following theorems, whose proof are ignored.

Theorem

If Y_1 and Y_2 are two solutions to a nonhomogeneous equation, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation.

Theorem

The solution to a nonhomogeneous equation can be written in the form of

$$y(t) = y_c(t) + y_p(t)$$

where y_c is the complementary (general) solution to the corresponding homogeneous equation and y_p is the particular solution to this nonhomogeneous equation.

Method of Undetermined Coefficients

Given the nonhomogeneous equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

we always use the method of undetermined coefficients to make an initial assumption about the form of the particular solution $Y(t)$. If $g(t)$ has the form of $P_m(t)e^{rt}[c_1 cos(kt) + c_2 sin(kt)]$, then we assume that

$$Y_p(t) = x^s[Q_m(t)e^{rt}cos(kt) + R_m(t)e^{rt}sin(kt)]$$

Here, P_m, Q_m and R_m are all polynomials of degree m.
Let's look at an example problem.

Method of Undetermined Coefficients

Given the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10cos(3x),\ y(0) = 1,\ y'(0) = 2$$

we can easily solve that $y_c(x) = c_1e^x + c_2e^{2x}$. Since $g(x) = 3e^{-x} - 10cos(3x)$, we assume that the particular solution is $y_p(x) = Ae^{-x} + Bcos(3x) + Csin(3x)$. Plug into the original equation to obtain that $A = \frac{1}{2}$, $B = \frac{7}{13}$ and $C = \frac{9}{13}$ directly. So

$$y(x) = c_1e^x + c_2e^{2x} - \frac{1}{2}e^{-x} - \frac{21}{13}sin(3x) + \frac{27}{13}cos(3x)$$

By the initial condition: $c_1 = -\frac{1}{2}$ $c_2 = \frac{6}{13}$

Variation of Parameters

Given the nonhomogeneous equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t)$$

we can directly try to determine the two functions u_1 and u_2 such that this homogeneous equation has the solution with the form of

$$y = u_1 y_1 + u_2 y_2$$

From this equation, we can obtain its first order and second order differentiation. To avoid the appearance of u_1'' and u_2'', we also assume that

$$u_1' y_1 + u_2' y_2 = 0$$

After substituting y'', y' and y and collecting the terms, we will derive that

$$u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2) + u_1'y_1' + u_2'y_2' = u_1'y_1' + u_2'y_2' = g$$

Variation of Parameters

Now, we can solve u_1' and u_2' separately. After the integration, we obtain that

$$y_p = -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dx$$

For example, given the equation $y'' + y = tan(x)$, then we first solve that $y_1 = cos(x)$ and $y_2 = sin(x)$. Hence,

$$u_1' = -sin(x)tan(x) = cos(x) - sec(x) \Rightarrow u_1 = sin(x) - ln \mid sec(x) + tan(x) \mid$$
$$u_2' = cos(x)tan(x) = sin(x) \Rightarrow -cos(x)$$

Now, we can find that the particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = -cos(x) ln \mid sec(x) + tan(x) \mid$$

High Order Linear Equation

The n-th order differential equations have the form of:

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + ... + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

When $G(t) = 0$, we say that this equation is homogeneous. The theorems we learnt for the 2nd order linear equations can also be applied to the high order linear equation. Let's have a brief summary.
The theorem of superposition is easy to understand and it is also the fundament of other theorems as well.

Theorem

Let $y_1, y_2, ..., y_n$ be n solutions to an n-th order homogeneous differential equation on the interval I. Then their linear combination

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

is also a solution to this equation on the interval I.

High Order Linear Equation

Another theorem that we have emphasized for several times is the theorem of existence and uniqueness.

Theorem

Suppose that $p_1, p_2, ..., p_n, f$ is continuous on the interval I including a. Then, for any given constants $b_0, b_1, ..., b_{n-1}$, the n-th order differential equation

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1} y' + p_n y = f$$

has a unique solution on the interval I, satisfying the initial condition

$$y(a) = b_0,\ y'(a) = b_1,\ ...,\ y^{(n-1)}(a) = b_{n-1}$$

Though, these two theorems are very easy, it is still necessary for us to follow a simple example.

High Order Linear Equation

For example, it is not difficult to verify that

$$y_1(x) = e^{-3x},\ y_2(x) = cos(2x),\ y_3(x) = sin(2x)$$

are three solutions to the 3rd order differential equation $y''' + 3y'' + 4y' + 12y = 0$. Therefore, from the theorem of superposition, we know that one of their linear combinations

$$y(x) = -3y_1(x) + 3y_2(x) - 2y_3(x) = -3e^{-3x} + 3cos(2x) - 2sin(2x)$$

is also a solution to this differential equation. Actually, this solution can be regarded as the particular solution satisfying the initial condition $y(0) = 0$, $y'(0) = 5$ and $y''(0) = -39$. According to the theorem of existence and uniqueness, this is the unique solution to this initial condition. From the curve, we find that when x is large enough, the solution will show a property of periodicity.

High Order Linear Equation

Now, let's discuss something about the Wronskian for n-th order differential equation. The definition is similar to what we learnt before.

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

We know that it is always necessary for us to find n linearly independent solutions to form a fundamental solution set. Fortunately, Wronskian can be applied to judge whether the n functions are linearly independent or not by the following theorem.

High Order Linear Equation

Theorem

Let $y_1, y_2, ..., y_n$ be n solutions to the following n-th order homogeneous differential equation on the interval I, where $p_1, p_2, ..., p_n, f$ is continuous on the interval I.

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1}y' + p_n y = f$$

If $y_1, y_2, ..., y_n$ are linearly dependent, then $W(y_1, y_2, ..., y_n) \equiv 0$ on the interval I. If $y_1, y_2, ..., y_n$ are linearly independent, then $W(y_1, y_2, ..., y_n) \neq 0$ for all $x \in I$.

By this theorem, we can find that the three solutions we obtained in the previous example are linearly independent, since $W(e^{-3x}, cos(2x), sin(2x)) = 26e^{-3x} \neq 0$.

High Order Linear Equation

The importance of finding the linearly independent solutions can be shown from the following theorem.

Theorem

Let $y_1, y_2, ..., y_n$ be n linearly independent solutions to the following n-th order homogeneous differential equation on the interval I, where $p_1, p_2, ..., p_n, f$ is continuous on the interval I.

$$y^{(n)} + p_1 y^{(n-1)} + ... + p_{n-1} y' + p_n y = f$$

Then any solution Y to this equation can be expressed as:

$$Y = c_1 y_1 + c_2 y_2 + ... + c_n y_n.$$

That is to say, from the previous discussion, all solutions to the 3rd order differential equation $y''' + 3y'' + 4y' + 12y = 0$ can be expressed as

$$y(x) = c_1 e^{-3x} + c_2 cos(2x) + c_3 sin(2x)$$

Linear Differential Operator

Before we continue talking about homogeneous equation with constant coefficients, let's first learn some concepts about linear differential operator. If we denote d/dx as D, then the linear differential operator can be defined as:

$$L_a(D) = a_n D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0$$

Here, we can regard D as a simple variable. For example, for the differential equation $y'' - (a+b)y' + aby = 0$, we can express it as $Ly = (D-a)(D-b)y = 0$. From the definition, it is easy to prove that

Theorem

$L_a(D)L_b(D) = L_b(D)L_a(D) = \sum a_i b_j D^{i+j}$

Linear Differential Operator

One more important property of the linear differential operator is exponential shift.

Theorem

$L_a(D)(ue^{rx}) = e^{rx}L_a(D+r)u$

Proof.

We first discuss the first order term. Notice that

$$(D-r)(ue^{rx}) = D(ue^{rx}) - r(ue^{rx}) = (Du)e^{rx} + u(re^{rx}) - r(ue^{rx}) = (Du)e^{rx}$$

The remaining steps can be done by mathematical induction.

Homogeneous Equation with Constant Coefficients

From now on, we will mainly focus on the high order homogeneous equations with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

The characteristic equation of this differential equation is:

$$a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0$$

Now, let's discuss several different situations according to the state of roots of this characteristic equation in detail.

Homogeneous Equation with Constant Coefficients

The simplest case is the situation when the characteristic equation has n distinct real roots, $r_1, r_2, ..., r_n$. Then , the general solution to the differential equation is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x}$$

For any real root to the characteristic equation, r_i, we know that

$$a_n r_i^n + a_{n-1} r_i^{n-1} + ... + a_1 r_i + a_0 = 0$$

We can multiply each side by $e^{r_i x}$ to obtain the original differential equation, since $\frac{d^k}{dx^x} e^{r_i x} = r_i^k e^{r_i x}$.

Homogeneous Equation with Constant Coefficients

Now, let's suppose that one of the roots of the original differential equation r has a multiplicity of k. For this case, the linear differential operator can be written as

$$L_a(D) = ...(D-r)^k...$$

Hence, the solution to the k-th order equation

$$(D-r)^k y = 0$$

is also the solution to the original differential equation as well. From the discussion in the previous case, we know that $y_1 = e^{rx}$ can be a solution to this differential equation. Using the thoughts of reduction of order, we want to try a solution with the form of $y_2 = uy_1$. By the property of exponential shift

$$(D-r)^k(ue^{rx}) = (D^k u)e^{rx}$$

That is to say, $D^k u$ is required to be zero.

Homogeneous Equation with Constant Coefficients

This is how we obtain that for such an r, the general solution should have a form of

$$y = (c_0 + c_1 x + ... + c_{k-1} x^{k-1}) e^{rx}$$

By now, we only consider the real roots. In fact, the sayings above also holds true for the complex roots. For example, if $r_1 = a + bi$ and $r_2 = a - bi$ are one pair of conjugated roots, then we can find a solution with the form of

$$y(x) = (c_1 + c_2) e^{ax} cos(bx) + i(C_1 - C_2) e^{ax} sin(bx)$$

We can find the first real solution by setting $c_1 = c_2 = \frac{1}{2}$ and the other real solution by setting $c_1 = -\frac{1}{2}i$ and $c_2 = \frac{1}{2}i$. An easy example may be the differential equation $y^{(4)} + 2y'' + y = 0$. It has a solution of the form $y = c_1 cos(t) + c_2 sin(t) + c_3 tcos(t) + c_4 tsin(t)$.

Method of Annihilator

The method of undetermined coefficients through "observing" still works now. It is recommended to review this concept by yourself. A new method of annihilator method will be introduced. This is also used to find the particular solution for a nonhomogeneous linear differential equation with constant coefficients.

Suppose that we are give a nonhomogeneous linear differential equation with constant coefficients (which can definitely expressed in the following form):

$$L_a(D)y = g(t)$$

Here, $g(t)$ can be a sum or product of exponential, polynomial or sinusoidal terms. We can first find the annihilator $A(D)$ for $g(t)$ such that $A(D)g(t) = 0$. Then we can first solve the equation

$$A(D)L(D)y = A(D)g(t) = 0$$

to get one form of solution y. The original particular solution has the form of $y - y_c$. The remaining work is to compare the coefficients.

Method of Annihilator

The following theorems are given without proof. But it is recommended to prove them by yourself.

Theorem

For $g(x) = P_n(t)e^{rt}$, *the corresponding annihilator is* $A(D) = [D - r]^{n+1}$.

Theorem

For $g(x) = P_n(t)e^{rt}[c_1 cos(kt) + c_2 sin(kt)]$, *the corresponding annihilator is* $A(D) = [(D - r)^2 + k^2]^{n+1}$.

Theorem

For $g(x) = g_1(x) + g_2(x)$, *the corresponding annihilator is* $A(D) = A_1(D)A_2(D)$.

For example, $D + 1$ is the annihilator of e^{-x}; $D^2 + 4$ is the annihilator of $sin(2x)$ and $cos(2x)$; $(D - 3)^2$ is the annihilator of xe^{3x}.

Method of Annihilator

Let's use the two different methods to solve the differential equation $y'' - 3y' = 8e^{3x} + 4sin(x)$.

By the method of undetermined coefficients, we should first guess that the solution has the form of $y = x^{s_1} C_0 e^{3x} + x^{s_2}[A_0 cos(x) + B_0 sin(x)]$. Since 0 and 3 are two roots for the corresponding characteristic equation, we know that $s_1 = 1$ and $s_2 = 0$. That is to say, the particular solution of this differential equation has the form of $y = C_0 xe^{3x} + A_0 cos(x) + B_0 sin(x)$.

By the method of annihilator, we should first find $A(n) = (D - 3)(D^2 + 1)$. Then we solve the equation $(D-3)(D^2+1)(D^2-3D)y = 0$ to obtain $y = C_1 + C_2 e^{3x} + C_3 xe^{3x} + C_4 cos(x) + C_5 sin(x)$. Notice that the original differential equation is $(D^2 - 3D)y = 0$, therefore $y_c = C_1 + C_2 e^{3x}$. That is to say, the particular solution of this differential equation has the form of $y = C_3 xe^{3x} + C_4 cos(x) + C_5 sin(x)$.

Step Function and Impulse Function

To deal with problems about jumping discontinuities, it is necessary for us to learn the unit step function. We can define this function as

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geqslant a \end{cases}$$

To represent the impulse nature in a very short time, it is necessary to introduce the impulse function. We can define this function as

$$\delta_a(t) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$

Also, we should notice that

$$\frac{d}{dt} u_a(t) = \delta_a(t)$$

Convolution Integral

If $f = 0$ and $g = 0$ holds true for $t < 0$, then the function h can be defined as the convolution of f and g with the following relation

$$h(t) = f * g = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

It is not difficult to find the commutative law, the distributive law and the associative law can be applied to this new operation as well. But notice that though $f * 0 = 0$ is always true, we cannot always derive that $f * 1 = f$.

Laplace Transform

The Laplace Transform can be used to solve linear ordinary differential equations. In contrast with the classical method of solving linear differential equations, this new method has the following features:
(1) The homogeneous equation and the particular integral of the solution of the differential equation are obtained in one operation.
(2) The Laplace transform converts the differential equation into an algebraic equation in s.
Given the function $f(t)$ defined for all $t \geqslant 0$, then the Laplace Transform of $f(t)$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t) dt,$$

satisfying the condition that

$$\int_0^\infty \mid f(t) e^{-\sigma t} \mid dt < \infty$$

if s is equal to $\sigma + \omega i$.

Laplace Transform

For example, if we define $f(t)$ to be $e^{\alpha t}$ for $t \geqslant 0$, then we will find that

$$F(s) = \frac{1}{s - \alpha}$$

Notice that the region of convergence is $s > \alpha$. Please remember the following table for future reference:

$f(t)$	$F(s)$	ROC	$f(t)$	$F(s)$	ROC
$\delta_0(t)$	1	All s	$u_0(t)$	$\frac{1}{s}$	$s > 0$
$t^n \;\; n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$s > 0$	$t^p \;\; p > -1$	$\frac{\int_0^\infty e^{-x} x^p dx}{s^{p+1}}$	$s > 0$
$sin(\alpha t)$	$\frac{\alpha}{s^2+\alpha^2}$	$s > 0$	$cos(\alpha t)$	$\frac{s}{s^2+\alpha^2}$	$s > 0$
$sinh(\alpha t)$	$\frac{\alpha}{s^2-\alpha^2}$	$s >\mid \alpha \mid$	$cosh(\alpha t)$	$\frac{s}{s^2-\alpha^2}$	$s >\mid \alpha \mid$

Properties of the Laplace Transform

The applications of the Laplace Transform in many instances are simplified by utilization of the properties of the transform. These properties are presented by the following theorems. Some proofs will be given.

Theorem

Multiplication by a Constant *Let k be a constant, and $F(s)$ be the Laplace transform of $f(t)$, then*

$$L\{kf(t)\} = kF(s)$$

Theorem

Sum and Difference *Let $F_1(s)$ and $F_2(s)$ be the Laplace Transform of $f_1(t)$ and $f_2(t)$, respectively. Then*

$$L\{f_1(t) \pm f_2(t)\} = F_1(s) + F_2(s)$$

Properties of the Laplace Transform

Theorem

Differentiation *Let $F(s)$ be the Laplace Transform of $f(t)$. Also, $f^{(i)}(t)$ denotes the i-th order derivative of $f(t)$ with respect to t. Then*

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - ... - f^{(n-1)}(0)$$

Theorem

Integration *The Laplace Transform of the first integral of $f(t)$ with respect to t is the Laplace Transform of $f(t)$ divided by s. In general,*

$$L\{\int_0^{t_1} \int_0^{t_2} ... \int_0^{t_n} f(t) d\tau_1 d\tau_2 ... d\tau_n\} = \frac{F(s)}{s^n}$$

Properties of the Laplace Transform

Theorem

Laplacian Differentiation *Let $F(s)$ be the Laplace Transform of $f(t)$. Also, $F^{(i)}(s)$ denotes the i-th order derivative of $F(s)$ with respect to s, evaluated at $t = 0$. Then*

$$F^{(n)}(s) = L\{(-t)^n f(t)\}$$

Theorem

Time Scaling *Let k be a constant, and $F(s)$ be the Laplace transform of $f(t)$, then*

$$L\{f(kt)\} = F(s/k)/k$$

Properties of the Laplace Transform

Theorem

Time Shifting *The Laplace Transform of $f(t)$ delayed by time T is equal to the Laplace Transform $f(t)$ multiplied by e^{-sT}, that is*

$$L\{f(t-T)u_T(t)\} = e^{-sT}F(s)$$

Theorem

Complex Shifting *The Laplace Transform of $f(t)$ multiplied by $e^{\mp\alpha t}$, where α is a constant, is equal to the Laplace Transform $F(s)$, with s replaced by $s \pm \alpha$, that is*

$$L\{e^{\mp\alpha t}f(t)\} = F(s \pm \alpha)$$

Properties of the Laplace Transform

Theorem

Real Convolution (Complex Multiplication) *Let $F_1(s)$ and $F_2(s)$ be the Laplace Transform of $f_1(t)$ and $f_2(t)$, respectively, and $f_1(t) = 0$, $f_2(t) = 0$ for $t < 0$, then*

$$L\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$$

Theorem

Complex Convolution (Real Multiplication) *There is also a dual relation to the real convolution theorem. Essentially, this theorem states that multiplication in the real t-domain is equivalent to convolution in the complex s-domain, that is*

$$L\{f_1(t)f_2(t)\} = F_1(s) * F_2(s)$$

Properties of the Laplace Transform

Theorem

Initial Value Theorem *Let $F(s)$ be the Laplace Transform of $f(t)$, then if the limits exists*

$$\lim_{t\to 0} = \lim_{s\to\infty} sF(s)$$

Theorem

Final Value Theorem *Let $F(s)$ be the Laplace Transform of $f(t)$, then if the limits exists*

$$\lim_{t\to\infty} = \lim_{s\to 0} sF(s)$$

Properties of the Laplace Transform

Now, some proofs of these properties will be given.

Proof.

Differentiation We only consider the situation when $n = 1$. For any higher order, we can just make use the mathematical induction. Let's first focus on the integral

$$\int_0^\infty e^{-st} f'(t)dt$$

Let $t_1, t_2, ..., t_k$ be the k points of discontinuity in the interval $0 \leqslant t \leqslant b$. Denote t_0 as 0. By part yields, we derive that

$$\int_0^b e^{-st} f'(t)dt = \sum_{i=1} k(e^{-st} f(t) \mid_{t_{i-1}}^{t_i} + s \int_{t_{i-1}}^{t_i} e^{-st} f(t)dt)$$

Properties of the Laplace Transform

Proof.

Since f is continuous, we can simplify this relation to

$$\int_0^b e^{-st} f'(t)dt = e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt$$

When $b \to \infty$, we obtain that

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Proof.

Laplacian Differentiation Similarly, we only need to consider the situation when $n = 1$.

$$F'(s) = \int_0^\infty f(t)\frac{d}{ds}e^{-st}dt = \int_0^\infty -tf(t)e^{-st}dt = L\{-tf(t)\}$$

Properties of the Laplace Transform

Proof.

Time Scaling

$$\int_0^\infty f(kt)e^{-st}dt = \frac{1}{k}\int_0^\infty f(\tau)e^{-s\tau/k}d\tau = \frac{1}{k}F(\frac{s}{k})$$

Proof.

Time Shifting

$$L\{f(t-T)u_T(t)\} = \int_0^\infty e^{-st}f(t-T)u_T(t)dt = \int_T^\infty e^{-st}f(t-T)u_T(t)dt$$

Substitute $t-T$ with τ to get

$$L\{f(t-T)u_T(t)\} = \int_0^\infty e^{-s(\tau+T)}f(\tau)d\tau = e^{-sT}F(s)$$

Properties of the Laplace Transform

Proof.

Initial/Final Value Theorem By the property of differentiation, we obtain

$$L\{f'(t)\} = sF(s) - f(0)$$

Take the limit for both side to get

$$\lim_{s\to\infty} L\{f'(t)\} = \lim_{s\to\infty} \int_0^\infty f'(t)e^{-st}dt = 0$$
$$\lim_{s\to 0} L\{f'(t)\} = \lim_{s\to 0} \int_0^\infty f'(t)e^{-st}dt = \lim_{t\to\infty} f(t) - f(0)$$

Hence, we conclude that

$$\lim_{s\to\infty} sF(s) - f(0) = 0 \Rightarrow \lim_{t\to 0} f(t) = f(0) = \lim_{s\to\infty} sF(s)$$
$$\lim_{s\to 0} sF(s) - f(0) = \lim_{t\to\infty} f(t) - f(0) \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Properties of the Laplace Transform

These useful properties also enable us to derive some other elementary Laplace Transforms.

Let first find the Laplace Transform of $f(t) = tsin(kt)$. We should first notice that $f(0) = 0$ and $f'(0) = 0$, so in this situation we can simply the property of differentiation to

$$L\{f''(t)\} = s^2 L\{f(t)\}$$

Since $f''(t) = 2kcos(kt) - k^2 tsin(kt)$, we have

$$\frac{2ks}{s^2 + k^2} - k^2 L\{tsin(kt)\} = s^2 L\{tsin(kt)\} \Rightarrow L\{tsin(kt)\} = \frac{2ks}{(s^2 + k^2)^2}$$

Properties of the Laplace Transform

In the previous example, we have a look at the power of "differentiation". Now, we will also see that "integral" can play an important role in those similar problems as well.

If we want to find the inverse Laplace Transform of $F(s) = \frac{1}{s^2(s-a)}$, it is natural to make use of the property of integration.

$$L^{-1}\{\frac{1}{s(s-a)}\} = \int_0^t L^{-1}\{\frac{1}{(s-a)}\}dt = \int_0^t e^{at}dt = \frac{e^{at}-1}{a}$$

Repeat the similar steps, we can derive that

$$L^{-1}\{\frac{1}{s(s-a)^2}\} = [\frac{1}{a}(\frac{1}{a}e^{at} - t)]_0^t = \frac{e^{at} - at - 1}{a^2}$$

Partial Fraction Expansion

When the Laplace Transform solution of a differential equation is a rational function in s, it can be written as

$$G(s) = Q(s)/P(s)$$

where $P(s)$ and $Q(s)$ are polynomials of s. It is assumed that the order of $P(s)$ in s is greater than that of $Q(s)$. If *alpha* is the root of $P(s) = 0$ with the multiplicity of n, then the corresponding partial fraction expansion for this term is

$$\frac{A_1}{s-\alpha} + \frac{A_2}{(s-\alpha)^2} + ... + \frac{A_n}{(s-\alpha)^n}$$

Similarly, if this root is a complex root $\alpha + \beta i$, then the corresponding partial fraction expansion for this term is

$$\frac{A_1 s + B_1}{(s-\alpha)^2 + \beta^2} + \frac{A_2 s + B_2}{[(s-\alpha)^2 + \beta^2]^2} + ... + \frac{A_n s + B_n}{[(s-\alpha)^2 + \beta^2]^n}$$

Partial Fraction Expansion

You may want to remember two more elementary inverse Laplace Transform for future reference. We have already proven the correctness of the first one. If you are interested, please try to prove the correctness of the second one by yourself.

$$L^{-1}\{\frac{s}{(s^2+k^2)^2}\} = \frac{tsin(kt)}{2k}$$
$$L^{-1}\{\frac{1}{(s^2+k^2)^2}\} = \frac{sin(kt) - ktcos(kt)}{2k^3}$$

Solution of Linear Ordinary Differential Equation

With the help of the basic knowledge and the theorems about the Laplace Transform, it is possible for us to solve some problems with this new method. To solve an initial value problem is one of the common problems we should care about. For example, if we are given a differential equation $x'' + 6x' + 34x = 0$ and the initial condition $x(0) = 3$, $x'(0) = 1$. Take the Laplace Transform of both sides to get:

$$[s^2X(s) - 3s - 1] + 6[sX(s) - 3] + 34X(s) = 0 \Rightarrow X(s) = \frac{3s + 19}{s^2 + 6s + 34}$$

By partial fraction expansion,

$$X(s) = \frac{3s + 9}{(s + 3)^2 + 5^2} + \frac{10}{(s + 3)^2 + 5^2}$$

By the property of complex shifting, $x(t) = e^{-3t}[3cos(5t) + 2sin(5t)]$

Solution of Linear Ordinary Differential Equation

Let's try a more complicated initial value problem $y^{(4)} + 2y'' + y = 4te^t$ with the initial condition $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$. By taking the Laplace Transform for both sides and partial fraction expansion, we obtain

$$Y(s) = \frac{4}{(s-1)^2(s^2+1)^2} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{s^2+1}$$

We can plug different values of s into the relation above to get several equations. Then we can obtain that $A = 1, B = -2, C = 2, D = 0, E = 2, F = 1$. By taking the inverse Laplace Transform to $Y(s)$, we get

$$y(t) = (t-2)e^t + (t+1)sin(t) + 2cos(t)$$

Solution of Linear Ordinary Differential Equation

The Laplace Transform can also be used to solve a system of linear equation. Suppose that we are given a system:

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40sin(3t) \end{cases}$$

with the initial condition $x(0) = x'(0) = y(0) = y'(0) = 0$. We can take the Laplace Transform for both sides and collect the terms:

$$\begin{cases} (s^2 + 3)X(s) - Y(s) = 0 \\ -2X(s) + (s^2 + 2)Y(s) = \frac{120}{s^2+9} \end{cases}$$

By the Cramer's Rule and partial fraction expansion, we can obtain:

$$\begin{cases} X(s) = \frac{5}{s^2+1} - \frac{8}{s^2+4} + \frac{3}{s^2+9} \Rightarrow x(t) = 5sint - 4sin2t + sin3t \\ Y(s) = \frac{10}{s^2+1} + \frac{8}{s^2+4} - \frac{18}{s^2+9} \Rightarrow y(t) = 10sint + 4sin2t - 6sin3t \end{cases}$$

Vector Space

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, subject to the ten rules below.

1. $\mathbf{u}+\mathbf{v}$ is in V
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
4. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
5. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
6. cu is in V
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
9. $(cd)\mathbf{u}=c(d\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

With these axioms, one can easily prove some simple facts.

Vector Space

Let's prove that $\mathbf{0}$ is unique. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u}+\mathbf{w}=\mathbf{w}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in V. In particular, $\mathbf{0}+\mathbf{w}=\mathbf{0}$. But $\mathbf{0}+\mathbf{w}=\mathbf{w}$, by Axiom 4. Hence, $\mathbf{w}=\mathbf{0}+\mathbf{w}=\mathbf{0}$. The same, we can prove that $-\mathbf{u}$ is unique as well. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u}+\mathbf{w}=\mathbf{0}$ for a vector $\mathbf{u}$ in V. Then add $-\mathbf{u}$ to both sides to derive:

$$(-\mathbf{u})+(\mathbf{u}+\mathbf{w})=(-\mathbf{u})+\mathbf{0}$$
$$(-\mathbf{u})+\mathbf{u}+\mathbf{w}=(-\mathbf{u})+\mathbf{0} \text{ By Axiom 3}$$
$$\mathbf{0}+\mathbf{w}=(-\mathbf{u})+\mathbf{0} \text{ By Axiom 5}$$
$$\mathbf{w}=-\mathbf{u} \text{ By Axiom 4}$$

In the lecture slide, there is a proof concluding that $0\mathbf{u}=\mathbf{0}$ for all $\mathbf{u}$ in V. We can also prove that $c\mathbf{0}=\mathbf{0}$ for all scalars c.

$$c\mathbf{0}=c(\mathbf{0}+\mathbf{0})=c\mathbf{0}+c\mathbf{0} \text{ By Axiom 4 and 7}$$

Add the negative of c$\mathbf{0}$ to both sides to obtain

$$c\mathbf{0}+(-c\mathbf{0})=c\mathbf{0}+c\mathbf{0}+(-c\mathbf{0})$$
$$c\mathbf{0}+(-c\mathbf{0})=c\mathbf{0}+[c\mathbf{0}+(-c\mathbf{0})] \text{ By Axiom 3}$$
$$\mathbf{0}=c\mathbf{0}+\mathbf{0} \text{ By Axiom 5}$$
$$\mathbf{0}=c\mathbf{0} \text{ By Axiom 4}$$

Subspace

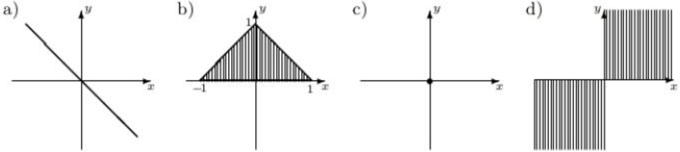
A subspace of a vector space V is a subset H of V with properties:

a. The zero vector of V is in H.

b. H is closed under vector addition.

c. H is closed under multiplication by scalars.

Specially, the set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as $\{\mathbf{0}\}$



For example, in the figures above, only (a) and (c) can be viewed as the subspace of $\mathbb{R}^2$. In (b), property b. and c. cannot be satisfied. In (d), property b. cannot be satisfied.

Subspace

Also, we should notice that the vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. However, the set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

is a subset of $\mathbb{R}^3$ that looks and acts like $\mathbb{R}^2$.
Here is an important theorem saying that

Theorem

If $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_2, ..., \mathbf{v}_p\}$ is a subspace of V.

Recall that Span$\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combination of $\mathbf{v}_1, ..., \mathbf{v}_p$.
The following example shows how to use this theorem.

Subspace

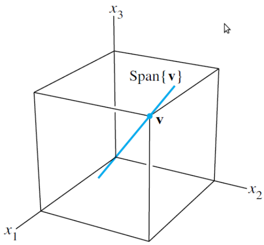
Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$, where a and b are arbitrary scalars. Show that H is a subspace of $\mathbb{R}^4$. From this description, we know that we can write any vector in H as

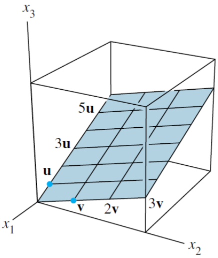
$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2$$

This calculation shows that H=Span$\{v_1, v_2\}$. Thus H is a subspace of $\mathbb{R}^4$ by the theorem above.

Subspace

Now, let's explain this concept from a view of geometry.
Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, and we visualize it as the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$, as shown in the left figure.
Similarly, if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ containing $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$, as shown in the right figure.





Subspace

Also, we should review how to use the echelon form of an augmented matrix to judge whether a vector is in the subspace spanned by other matrix. For example, show that $\mathbf{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1, \mathbf{v}_2\mathbf{v}_3$, where

$$\mathbf{w} = \begin{bmatrix} -9 \\ 7 \\ 4 \\ 8 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 7 \\ -4 \\ -2 \\ 9 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 5 \\ -1 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -9 \\ 4 \\ 4 \\ -7 \end{bmatrix}$$

Here, we should first obtain the reduced echelon form of the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}]$.

$$\begin{bmatrix} 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -2 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7.5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\mathbf{w} = 7.5\mathbf{v}_1 + 3textbfv_2 + 5.5\mathbf{v}_3$. The corresponding MATLAB command is "rref()". You can try this problem by yourself.

Subspace

If it is necessary for you to review the row operation, the following simple example will help you.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two examples about the row reduction in Section 7.3 of your textbook are also very easy to understand.

Linearly Independent Sets and Bases

A set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in V is said to be linearly independent if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p = \mathbf{0}$$

has only trivial solution, $c_1 = 0, ..., c_p = 0$.
Let H be a subspace of a vector space V. A set of vector B=$\{\mathbf{b}_1, ..., \mathbf{b}_p\}$ in V is a basis of H if
a. B is a linearly independent set
b. H=Span$\{\mathbf{b}_1, ..., \mathbf{b}_p\}$
In fact, a basis is an efficient spanning set that contains no unnecessary vectors and can be constructed from a spanning set by discarding unneeded vectors.

Theorem

Let $S=\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ be a set in V, and let $H=Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$. If one of the vectors in S, $\mathbf{v}_k$ is a linear combination of the remaining vectors in S, then the set formed by removing $\mathbf{v}_k$ still spans H. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Linearly Independent Sets and Bases

We divide the proof into two parts. For the first half of saying:

Proof.

By rearranging the list of vectors in S, for convenience, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1, ...\mathbf{v}_p$

$$\mathbf{v}_p = a_1\mathbf{v}_1 + ... + a_{p-1}\mathbf{v}_{p-1}$$

Given any $\mathbf{x}$ in H, we may write

$$\mathbf{x} = a_1\mathbf{v}_1 + ... + a_p\mathbf{v}_p$$

From these two equations, it is easy to see that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_{p-1}$. Thus, $\{\mathbf{v}_1, ..., \mathbf{v}_{p-1}\}$ spans H.

Linearly Independent Sets and Bases

The following is the proof for the second half:

Proof.

If the original spanning set S is linearly independent, then it is already a basis for F. Otherwise, one of the vectors in S depends on the others can be deleted by the first half of this theorem. As long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and is a basis of H. If the spanning set is eventually reduced to one vector, that vector will be nonzero because H$\neq \{\mathbf{0}\}$.

We can regard a basis as a spanning set that is as large as possible or a linearly independent set that is as large as possible. The following example will illustrate this point for you.

Linearly Independent Sets and Bases

The following sets are all sets in $\mathbb{R}^3$. The left one is linearly dependent but does not span $\mathbb{R}^3$. The right one spans $\mathbb{R}^3$ but is linearly dependent. Only the middle set can be used as the basis of $\mathbb{R}^3$.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Now, let's try some problems and recall some useful methods about linearly independent sets and bases.

Linearly Independent Sets and Bases

We have learnt that the Wronskian can be used to judge whether several equations are linearly independent or not. For example, we can check that $y_1 = x$, $y_2 = x\ln(x)$ and $y_3 = x^2$ are solutions to the third order differential equation

$$x^3y^{(3)} - x^2y'' + 2xy' - 2y = 0 \ (x > 0)$$

Their Wronskian is calculated as

$$\mathrm{W}=\begin{vmatrix} x & x\ln(x) & x^2 \\ 1 & 1+\ln(x) & 2x \\ 0 & 1/x & 2 \end{vmatrix} = x$$

Since this value is not zero for all x on the interval $(0, \infty)$, these solutions are linearly independent.

Linearly Independent Sets and Bases

Not only the Wronskian but also the simple determinant can be applied in some of these kinds of problems as well. Consider whether $P_1(x) = x^2 - 1$, $P_2(x) = x$ and $P_3(x) = -x^2 + 2x + 3$ are linearly independent in the vector space $\mathbb{P}_2$. It is enough to check the determinant

$$\text{D}=\begin{vmatrix} -1 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = -2$$

Since this value is nonzero, we conclude that they are linearly independent.
Recall that the determinant of an $n \times n$ matrix A can be computed by a cofactor expansion.

$$\det(\text{A})=a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$
$$\det(\text{A})=a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$
$$C_{ij} = (-1)^{i+j}\det A_{ij}$$

By the way, the command to calculate the determinant in MATLAB is “det()”.

Inner Product

This subsection is just a quick review. Since all the materials here has already been covered in Vv255 and can be easily understood.
First we should recall that the inner product of two vectors are defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} x_k y_k$$

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then
a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
d. $\mathbf{u} \cdot \mathbf{u} \geqslant 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$
The inner product can be used to calculate the length of a vector

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Thus, the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Orthogonal Sets

A set of vector $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal if each pair of distinct vectors from the set is orthogonal, that is $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$. Thus, an orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set. The following two theorems suggest why an orthogonal basis is much nicer than other bases.

Theorem

If $S = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Theorem

Let $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination are given by

$$\mathbf{y} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p \ \ c_i = \frac{\mathbf{y} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

Orthogonal Sets

To prove these two theorems:

Proof.

If $\mathbf{0} = c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p$ for some scalar $c_1, ..., c_p$ then

$$\begin{aligned} 0 = \mathbf{0} \cdot \mathbf{u}_1 &= (c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= (c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 ... + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + ... + c_p(\mathbf{u}_p \cdot \mathbf{u}_1) \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) \end{aligned}$$

Therefore, c_1 is equal to zero. We can get the similar conclusions for other scalars in the same way.

For the second proof, we just change $\mathbf{0}$ to $\mathbf{y}$ to draw the final conclusion.

Inner Product Space

We have already introduced four properties of the inner product for $\mathbb{R}^n$. For other spaces, we need analogues of the inner product with the same properties. In fact, an inner product on a vector space V is a function that, to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ and satisfies the following axioms, for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and all scalars c:

a. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
b. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
c. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
d. $\langle \mathbf{u}, \mathbf{u} \rangle \geqslant 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$

Similarly, the inner product can be used to calculate the length of a vector

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

The same, the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ are defined

$$\text{dist}(\mathbf{u},\mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Inner Product Space

For the vector space $\mathbb{R}^2$, set

$$\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2$$

We can check that this defines an inner product on $\mathbb{R}^2$. Actually, inner products similar to this one can be defined in $\mathbb{R}^n$. They arise naturally in connection with "weighted least-squares" problems.
Let $\mathbf{y}$ be a vector of n observations, $y_1, ..., y_n$, and suppose we wish to approximate $\mathbf{y}$ by $\hat{\mathbf{y}}$ that belongs to some specified subspace of $\mathbb{R}^n$. Denote the entries in $\hat{\mathbf{y}}$ by $\hat{y}_1, ..., \hat{y}_n$. Then "the sum of squares for error" in approximating $\mathbf{y}$ by $\hat{\mathbf{y}}$ is

$$\text{SS(E)}=(y_1 - \hat{y}_1)^2 + ... + (y_n - \hat{y}_n)^2$$

If the weights are denoted by $w_1^2, ..., w_n^2$, then the "weighted sum of squares for error" is

$$\text{WSS(E)}=w_1^2(y_1 - \hat{y}_1)^2 + ... + w_n^2(y_n - \hat{y}_n)^2$$

Inner Product Space

Let's look at the other example. Let $t_0, ..., t_n$ be distinct real numbers. For p and q in $\mathbb{P}_n$, define

$$\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + ... + p(t_n)q(t_n)$$

We can also check that this satisfies the four properties listed in the previous slide and defines a inner product on $\mathbb{P}_n$.

Gram-Schmidt Process

The Gram-Schmidt Process is a simple algorithm for producing an orthogonal basis for any nonzero subspace of $\mathbb{R}^n$. Given a basis $\{\mathbf{x}_1, ..., \mathbf{x}_p\}$ for a subspace W of $\mathbb{R}^n$, then this algorithm will produce $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ as an orthogonal basis for W.

Algorithm

procedure *Gram-Schmidt*($\{\mathbf{x}_1, ..., \mathbf{x}_p\}$,$\{\mathbf{v}_1, ..., \mathbf{v}_p\}$)
  for $i := 1$ **to** p
    $\mathbf{v}_i := \mathbf{x}_i$
    for $k := 1$ **to** $i - 1$
      $\mathbf{v}_i := \mathbf{v}_i$ - $(\langle \mathbf{x}_i, \mathbf{v}_k \rangle / \langle \mathbf{v}_k, \mathbf{v}_k \rangle)\mathbf{v}_k$;

There is a theorem showing that any nonzero subspace W of $\mathbb{R}^n$ has an orthogonal basis, because an ordinary basis $\{\mathbf{x}_1, ..., \mathbf{x}_p\}$ is always available. This theorem is based on the dimension of a vector space, so let's ignore it here. You can just believe that this kind of orthogonal basis always exists.
The correctness of this algorithm is based on the Orthogonal Decomposition Theorem, we will also ignore the proof of its correctness.

Gram-Schmidt Process

Let V be $\mathbb{P}_4$ with the inner product in the previous slides:

$$\langle p, q\rangle = p(-2)q(-2) + p(-1)q(-1) + ... + p(2)q(2) \text{ for } p \text{ and } q \text{ in } \mathbb{P}_4$$

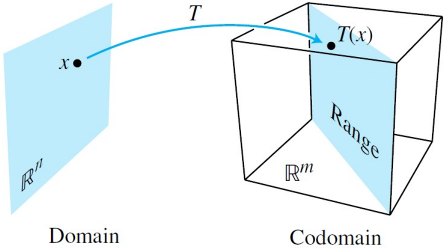
involving evaluation of polynomials at -2,-1,0,1 and 2, and view W as $\mathbb{P}_2$, a subspace of V. We can produce an orthogonal basis for $\mathbb{P}_2$ by applying the Gram-Schmidt Process to the polynomials $1, t, t^2$.
The first step of this algorithm shows that we directly derive that $p_0(t) = 1$. Since, we know that $\langle t, p_0(t)\rangle = 0$, we can obtain that $p_1(t) = t$. To get p_2, we first find:

$$\langle t^2, p_0\rangle = \langle t^2, 1\rangle = 10$$
$$\langle t^2, p_1\rangle = \langle t^2, t\rangle = 0$$
$$\langle p_0, p_0\rangle = 5$$

Therefore, $p_2(t) = t^2 - 2p_0(t) = t^2 - 2$.

Linear Transformation

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T. In the future, we will mainly talk about the matrix transformation.



Please use the figure above to make the concepts about "domain", "codomain", "range" clear.

Linear Transformation

Linear transformation preserves the operations of vector addition and scalar multiplication. That is to say:

Theorem

A transformation T is linear if
(1) $T(\mathbf{u}+\mathbf{v}) = T\mathbf{u} + T\mathbf{v}$
(2) $T(c\mathbf{u}) = cT\mathbf{u}$

It will not be difficult to lead the following useful facts:

Theorem

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$
$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c, d.

Linear Transformation

Whenever a linear transformation T arises geometrically or is described in words, we usually want a "formula" for T. It will be shown that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix represented as

$$A = [T(\mathbf{e}_1) \dots T(\mathbf{e}_n)]$$

Here, A is called as the standard matrix for the linear transformation T. Now, let's prove the correctness of this theorem.

Linear Transformation

Proof.

Write $\mathbf{x} = \mathbf{I}_n\mathbf{x} = [\mathbf{e}_1 ... \mathbf{e}_n]\mathbf{x} = x_1\mathbf{e}_1 + ... x_n\mathbf{e}_n$, and use the linear of T to compute

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + ... + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + ... + x_n T(\mathbf{e}_n) \\ &= [T(\mathbf{e}_1) ... T(\mathbf{e}_n)][x_1 ... x_n]^T \\ &= A\mathbf{x} \end{aligned}$$

The existence is proven by now. To verify the uniqueness, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B, and let A be the standard matrix for T. By definition, $A = [T(\mathbf{e}_1) ... T(\mathbf{e}_n)]$. However, by matrix-vector multiplication, $T(\mathbf{e}_j) = B\mathbf{e}_j = \mathbf{b}_j$, the j-th column of B. So, $A = B$, which derives the uniqueness of A.

Linear Transformation

For example, if we want to find the standard matrix for the linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal x-axis, we just need to care about the two points $(1, 0)$ and $(0, 1)$. Then it will not be difficult for us to find that the corresponding standard matrix is

$$\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Kernel and Range of Linear Transformation

Let T be a linear transformation from a vector space V into a vector space W, then the kernel of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W). The range of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

If T happens to arise as a matrix transformation, then the kernel and the range of T are just the null space and the column space of A.

The null space of an $m \times n$ matrix A, written as Nul$A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul}A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

The column space of an $m \times n$ matrix A, written as Col$A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 ... \mathbf{a}_n]$, then

$$\text{Col}A = \text{Span}\{\mathbf{a}_1 ... \mathbf{a}_n\}$$

Dimension, Rank and Nullity

If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. It can be concluded that the dimension of Nul$A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col$A$ is the number of pivot columns in A.
For example, given the following matrix, we can reduce it to the echelon form:

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, there are three free variables (x_2, x_4, x_5), dimNul$A$ is 3. Also, since there are two pivot columns, dimCol$A$ is 2.

Dimension, Rank and Nullity

Actually, the dimension of the range of a linear transformation can be defined as the rank of this transformation and the dimension of the kernel of a linear transformation can be define as the nullity of this transformation. Then result of the previous example problem can also be illustrated by the following theorem

Theorem

Let $T : V \to W$ be the linear transformation from a vector space V into a vector space W. Then

$$rankT{+}nullityT{=}dimT$$

Let's use the linear transformation $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ as an example. Since ran$D$ is $\mathbb{P}_{n-1}$, rank$D$ is dim$\mathbb{P}_{n-1}$, which is n. Since ker$D$ is the set of all constant polynomials, nullity$D$ is 1. Finally, we observe that dimP_n is $n + 1$.
Before we go through another example, let's make some efforts convince ourselves that this theorem is correct.

Dimension, Rank and Nullity

Proof.

Let the basis of $\ker T$ to be $\{\mathbf{u}_1, ..., \mathbf{u}_r\}$. Let the basis of V to be $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{w}_1, ..., \mathbf{w}_n\}$. Then it is enough to prove that the dimension of $\operatorname{ran} T$ is n.
For an arbitrary vector $\mathbf{v}$ in V, we have:

$$\begin{aligned} &\mathbf{v} = a_1\mathbf{u}_1 + ... + a_r\mathbf{u}_r + b_1\mathbf{w}_1 + ... + b_n\mathbf{w}_n \\ \Rightarrow &T\mathbf{v} = a_1 T\mathbf{u}_1 + ... + a_r T\mathbf{u}_r + b_1 T\mathbf{w}_1 + ... + b_n T\mathbf{w}_n \\ \Rightarrow &T\mathbf{v} = b_1 T\mathbf{w}_1 + ... + b_n T\mathbf{w}_n \end{aligned}$$

That is to say $\operatorname{ran} T$ =Span$\{T\mathbf{w}_1, ..., T\mathbf{w}_n\}$. Now, we just need to show that $\{T\mathbf{w}_1, ..., T\mathbf{w}_n\}$ is a linearly independent set.

Dimension, Rank and Nullity

Proof.

Consider the linear combination

$$c_1 T\mathbf{w}_1 + ... + c_n T\mathbf{w}_n = T(c_1\mathbf{w}_1 + ... + c_n\mathbf{w}_n) = 0$$

That is to say

$$c_1\mathbf{w}_1 + ... + c_n\mathbf{w}_n \in \ker T \Rightarrow c_1\mathbf{w}_1 + ... + c_n\mathbf{w}_n = d_1\mathbf{u}_1 + ... + d_r\mathbf{u}_r$$

Since $\{\mathbf{u}_1, ..., \mathbf{u}_r, \mathbf{w}_1, ..., \mathbf{w}_n\}$ forms the basis of V, all c_i and d_i should be zero. Therefore, we derive that $\{T\mathbf{w}_1, ..., T\mathbf{w}_n\}$ is a linearly independent set.

Let's consider the linear transformation $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by

$$T(P_n(x)) = x^2 P_n''(x) - 2xP_n'(x) + P(x)$$

What is $\operatorname{rank} T$ and $\operatorname{nullity} T$?

Dimension, Rank and Nullity

We can find the general solution to

$$T(f(x)) = 0$$

to obtain $f(x) = Ae^{\alpha x} + Be^{\beta x}$, where $\alpha, \beta = (5 \pm \sqrt{5})/2$. Since we want $f(x)$ to be a polynomial, $f(x)$ can only be zero. That is to say, $\ker T$ is $\{0\}$. By the rank-nullity theorem, $\operatorname{rank} T$ should be $n + 1$. That is to say, for any $Q_n(x)$ in $\mathbb{P}_n$, we can find a unique polynomial solution to

$$T(P_n(x)) = Q_n(x)$$

Eigenvalues and Eigenvectors

A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.
When an eigenvalue is known, we can use row reduction to find the corresponding eigenvector. For example, 2 is an eigenvalue of the following matrix

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Actually, what we should do is just to find the basis of the corresponding eigenspace.

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Given a particular $n \times n$ matrix, we can find all its eigenvalues with the help of the characteristic equation

$$\det(A - \lambda I)=0$$

For example, we can analyze the long-term behavior of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $(k = 0, 1, ...)$, with $\mathbf{x}_0 = [.6\ .4]^T$, if A is defined as

$$A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$$

From the characteristic equation $0 = \det(A-\lambda I) = (.95-\lambda)(.97-\lambda)-(.03)(.05)$, we derive that $\lambda_1 = 1$ and $\lambda_2 = .92$. By row deduction, we can get the corresponding eigenvectors as $\mathbf{v}_1 = [3\ 5]^T$ and $\mathbf{v}_2 = [1\ -1]^T$. Since $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we can get $c_1 = .125$ and $c_2 = .225$.
In general, $\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2$. When $k \to \infty$, we have $\mathbf{x}_k \to [.375\ .625]^T$.

Diagonalization

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

For example, suppose that we are given the 3×3 matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

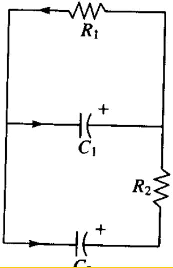
It is not difficult to find its eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Then three linearly independent eigenvectors should be found. Fortunately, we find that $\mathbf{v}_1$ corresponding to λ_1 and $\mathbf{v}_2$, $\mathbf{v}_3$ corresponding to λ_2.

Diagonalization

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Finally, we can determine P as $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, and D as the diagonal matrix with D_{ii} equal to λ_i.

You may wonder why we should learn this. Let's briefly go through the following example. The circuit in the following figure can be described by a differential equation, where $v_1(t)$ and $v_2(t)$ are the voltages across the two capacitors at time t.



Diagonalization

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2 C_1) \\ 1/(R_2 C_2) & -1/(R_2 C_2) \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

Given the data, $R_1 = 1\Omega$, $R_2 = 2\Omega$, $C_1 = 1F$ and $C_2 = .5F$, we can set

$$A = \begin{bmatrix} -1.5 & .5 \\ 1 & -1 \end{bmatrix}$$

Then, we can express the desired voltages as

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

Here, λ_1 and λ_2 are the eigenvalues of A. Furthermore, $\mathbf{v}_1$ and $\mathbf{v}_2$ are their corresponding eigenvectors.

By the way, you can make use of the "eig()" command in MATLAB to diagonalize a matrix.

Matrix Exponential

Recall that the scalar exponential function $exp(at)$ can be represented by the power series

$$exp(at) = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}$$

which converges for all t. Let's now replace the scalar a by the $n \times n$ constant matrix A and consider the corresponding series

$$exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$$

It is not easy to get the matrix exponential $exp(A)$ from this definition. It is better to use diagonalization so that we can find the relation that

$$exp(A) = exp(PDP^{-1}) = Pexp(D)P^{-1}$$

Matrix Exponential

The derivation is quite straightforward.

$$\begin{aligned} exp(A) &= I + PDP^{-1} + \frac{1}{2}PDP^{-1}PDP^{-1} + \frac{1}{6}PDP^{-1}PDP^{-1}PDP^{-1} + ... \\ &= I + PDP^{-1} + \frac{1}{2}PD^2P^{-1} + \frac{1}{6}PD^3P^{-1} + ... \\ &= P(I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + ...)P^{-1} \\ &= Pexp(D)P^{-1} \end{aligned}$$

Notice that the corresponding command for matrix exponential in MATLAB is "expm()", instead of "exp()".

Hermitian Matrix

A matrix A is called to be Hermitian or self-adjoint when $A^* = A$; that is, $\overline{a_{ji}} = a_{ij}$. Hermitian matrices include as a subclass real symmetric matrices. Hermitian matrices always have the following properties:
(1) All eigenvalues are real.
(2) There always exists a full set of n linearly independent mutually orthogonal eigenvectors.
(3) If $A = PDP^{-1}$, then $P^{-1} = P^*$.

Basic Theory

The system of n first order differential equations has the form of

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + ... + p_{1n}(t) + g_1(t) \\ x_2' &= p_{21}(t)x_1 + ... + p_{2n}(t) + g_2(t) \\ &\;... \\ x_n' &= p_{n1}(t)x_1 + ... + p_{nn}(t) + g_n(t) \end{aligned}$$

To discuss it more effectively, we write it in matrix notation as

$$\mathbf{x}' = P\mathbf{x} + \mathbf{g}$$

Here, we assume that P and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$. When $\mathbf{g} = \mathbf{0}$, this system is said to be homogeneous. Several useful theorems will be given without proof. The proof is similar to what we discussed the ordinary differential equations, and you can find the proof in your textbook.

Basic Theory

Theorem

If vector functions $\mathbf{x}_1, ..., \mathbf{x}_n$ *are linearly independent solutions of the system showing in the previous slides for each point in the interval* $\alpha < t < \beta$, *then each solution of this system can be expressed as a linear combination of* $\mathbf{x}_1, ..., \mathbf{x}_n$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + ... + c_n\mathbf{x}_n$$

in exactly one way.

Theorem

If $\mathbf{x}_1, ..., \mathbf{x}_n$ *are solutions to the system showing in the previous slides on the interval* $\alpha < t < \beta$, *then in this interval* $W[\mathbf{x}_1, ..., \mathbf{x}_n]$ *either is identically zero or else never vanishes.*

Basic Theory

Theorem

Let $\mathbf{x}_1, ..., \mathbf{x}_n$ be the solutions to the system showing in the previous slides, satisfying the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{e}_1, ..., \mathbf{x}_n(t_0) = \mathbf{e}_n$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}_1, ..., \mathbf{x}_n$ form a fundamental set of solutions for this system.

For a homogeneous system of differential equations, we should seek solutions of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

We will discuss this according to the possibilities of the eigenvalues of A in the system

$$\mathbf{x}' = A\mathbf{x}$$

Saddle Point

Let A be the following matrix

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = -1 \; \lambda_2 = 2$$

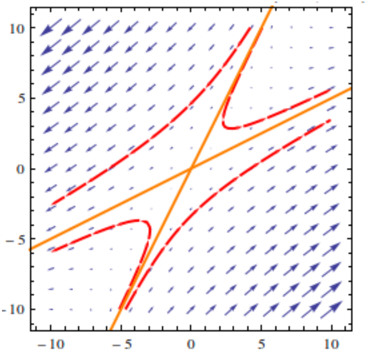
The corresponding eigenvectors are

$$\mathbf{v}_1 = [1 \; 2]^T$$
$$\mathbf{v}_2 = [2 \; 1]^T$$

The general solution is then $\mathbf{x} = c_1\mathbf{v}_1 e^{-t} + c_2\mathbf{v}_2 e^{2t}$. To sketch the trajectories, set $c_2 = 0$ to get $x_1 = c_1 e^{-t}$ and $x_2 = 2c_1 e^{-t}$. Thus, one asymptote is given as $x_2 = 2x_1$. Similarly, set $c_1 = 0$ to get the second asymptote as $x_2 = 0.5x_1$. As long as $c_2 \neq 0$, all solutions will be asymptotic to $x_2 = 0.5x_1$ as $t \to \infty$

Saddle Point

For this case, where eigenvalues are real and of opposite signs, the origin is called a saddle point. A saddle point is always unstable because almost all trajectories depart from the origin as t increases, although there are still some of the trajectories approaching this point.



Nodal Source

Let A be the following matrix

$$\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = 0.5 \ \lambda_2 = 2$$

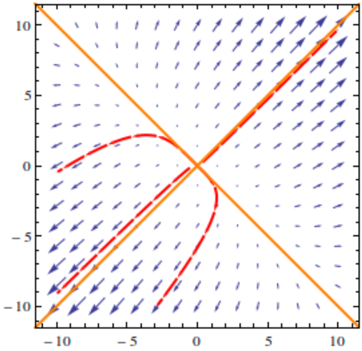
The corresponding eigenvectors are

$$\mathbf{v}_1 = [-1 \ 1]^T$$
$$\mathbf{v}_2 = [1 \ 1]^T$$

The general solution is then $\mathbf{x} = c_1 \mathbf{v}_1 e^{0.5t} + c_2 \mathbf{v}_2 e^{2t}$. To sketch the trajectories, set $c_2 = 0$ to get one asymptote is given as $x_2 = -x_1$. Similarly, set $c_1 = 0$ to get the second asymptote as $x_2 = x_1$.

Nodal Source

For this case, where eigenvalues are real, different and of positive signs, the origin is called a nodal source. A nodal source is always unstable because all trajectories depart from the origin as t increases.



Nodal Sink

Let A be the following matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = -1 \ \lambda_2 = -3$$

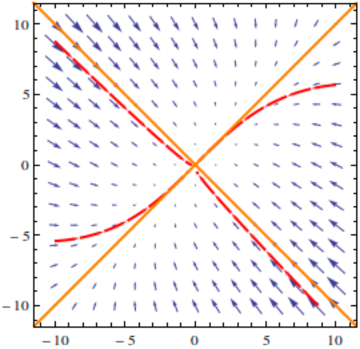
The corresponding eigenvectors are

$$\mathbf{v}_1 = [1\ 1]^T$$
$$\mathbf{v}_2 = [1\ -1]^T$$

The general solution is then $\mathbf{x} = c_1\mathbf{v}_1 e^{-t} + c_2\mathbf{v}_2 e^{-3t}$. To sketch the trajectories, set $c_2 = 0$ to get one asymptote is given as $x_2 = x_1$. Similarly, set $c_1 = 0$ to get the second asymptote as $x_2 = -x_1$.

Nodal Sink

For this case, where eigenvalues are real, different and of negative signs, the origin is called a nodal sink. A nodal sink is always asymptotically stable because almost all trajectories approach the origin along one of the asymptotes as t increases.



Improper Nodal Source

Let A be the following matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = 2 \; \lambda_2 = 2$$

With the geometric multiplicity of one, the corresponding eigenvector is

$$\mathbf{v} = [1 \; -1]^T$$

Thus one solution of this system is $\mathbf{x}_1 = c_1\mathbf{v}e^{2t}$. Based on what we learnt when solving second order linear equations, it may be natural to attempt to find a second solution of the form

$$\mathbf{x} = \mathbf{v}te^{2t}$$

Improper Nodal Source

However, if we plug it into the given system, we will obtain that

$$2\mathbf{v}te^{2t} + \mathbf{v}e^{2t} = A\mathbf{v}te^{2t}$$

Since this relation must be satisfied for all t, it is necessary for the coefficients of te^{2t} and e^{2t} both to be zero. From the term in e^{2t}, we find that

$$\mathbf{v}{=}\mathbf{0}$$

Hence, there is no nonzero solution of the given system with this form. Since we find that both te^{2t} and e^{2t} terms are included, it appears that in addition to $\mathbf{v}te^{2t}$, the second solution must contain a term of the form $\mathbf{u}te^{2t}$. Here, we call $\mathbf{u}$ as the generalized eigenvector corresponding to the repeated eigenvector. This time, we assume that

$$\mathbf{x} = \mathbf{v}te^{2t} + \mathbf{u}e^{2t}$$

so that we can get

$$2\mathbf{v}te^{2t} + (\mathbf{v} + 2\mathbf{u}) = A(\mathbf{v}te^{2t} + \mathbf{u}e^{2t})$$

Improper Nodal Source

Equating coefficients of te^{2t} and e^{2t} on each side, we derive that

$$(A-2I)\mathbf{v}=\mathbf{0}$$
$$(A-2I)\mathbf{u}=\mathbf{v}$$

Though $\det(A-2I)$ is zero, we can show that the second equation is always solvable. For this problem, we will derive that $u_1+u_2+1=0$. That is to say,

$$\mathbf{u}=[0\ -1]^T+k[1\ -1]^T$$

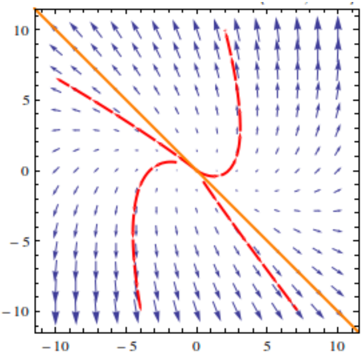
Further, we can obtain $\mathbf{x}_2$. Notice that the last term is merely a multiple of $\mathbf{x}_1$, so we can ignore it in fact.

$$\mathbf{x}_2=[1\ -1]^T te^{2t}+[0\ -1]^T e^{2t}+k[1\ -1]^T e^{2t}\Rightarrow \mathbf{x}_2=[1\ -1]^T te^{2t}+[0\ -1]^T e^{2t}$$

The general solution is then $\mathbf{x}=c_1\mathbf{x}_1+c_2\mathbf{x}_2$. As $t\to\infty$, each trajectory is asymptotic to a line $x_2=-x_1$.

Improper Nodal Source

For this case, where eigenvalues are real, repeated and of positive signs, the origin is called an improper nodal source. An improper nodal source is unstable.



Improper Nodal Sink

Let A be the following matrix

$$\begin{bmatrix} -2 & 1 \\ -4 & -6 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = -4 \ \lambda_2 = -4$$

With the geometric multiplicity of one, the corresponding eigenvector is

$$\mathbf{v} = [-1 \ 2]^T$$

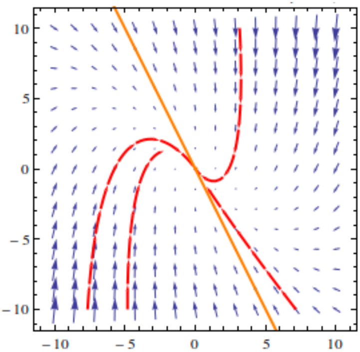
Similar, we can get a generalized eigenvector as well. Here, the details will be ignored.

$$\mathbf{u} = [0 \ -1]^T + k\mathbf{v} \Rightarrow \mathbf{u} = [0 \ -1]^T$$

The general solution is then $\mathbf{x} = c_1\mathbf{v}e^{-4t} + c_2(\mathbf{v}t + \mathbf{u})e^{-4t}$. As $t \to \infty$, all trajectories approach the origin.

Improper Nodal Sink

For this case, where eigenvalues are real, repeated and of negative signs, the origin is called an improper nodal sink. An improper nodal sink is asymptotically stable.



Spiral Source

Let A be the following matrix

$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = 1 + 2i \ \lambda_2 = 1 - 2i$$

The eigenvector corresponding to λ_1 is

$$\mathbf{v}_1 = [1 \ 1 - i]^T$$

Thus one solution of this system is $\mathbf{x}_1 = c_1 \mathbf{v}_1 e^{(1+2i)t}$. To find real-valued solutions, we take the real and imaginary parts, respectively of $\mathbf{x}_1$. Thus,

$$\mathbf{x}_1 = [1 \ 1 - i]^T e^t (cos(2t) + isin(2t))$$

That is to say,

$$\mathbf{x}_1 = e^t [cos(2t) \ cos(2t) + sin(2t)]^T + ie^t [sin(2t) \ sin(2t) - cos(2t)]^T$$

Spiral Source

We similar obtain $\mathbf{x}_2$ as

$$\mathbf{x}_2 = e^t[cos(2t)\ cos(2t) + sin(2t)]^T - ie^t[sin(2t)\ sin(2t) - cos(2t)]^T$$

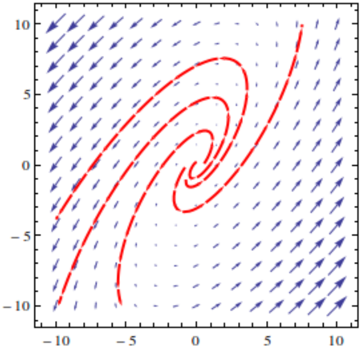
From what we learnt when solving the second order linear equation, it is not difficult to claim that we can get a real-valued solution

$$\mathbf{x} = c_1 e^t[cos(2t)\ cos(2t) + sin(2t)]^T + c_2 e^t[sin(2t)\ sin(2t) - cos(2t)]^T$$

All trajectories depart from the origin in this case. The trajectories will make infinitely many circuits about the origin, due to the sine and cosine factors.

Spiral Source

For this case, where eigenvalues are complex, and of positive real parts, the origin is called a spiral source. A spiral source is always unstable.



Spiral Sink

Let A be the following matrix

$$\begin{bmatrix} -0.5 & 1 \\ -1 & -0.5 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = -0.5 + i \ \lambda_2 = -0.5 - i$$

The eigenvector corresponding to λ_1 is

$$\mathbf{v}_1 = [1 \ i]^T$$

Thus one solution of this system is $\mathbf{x}_1 = c_1 \mathbf{v}_1 e^{(-0.5+i)t}$. To find real-valued solutions, we take the real and imaginary parts, respectively of $\mathbf{x}_1$. Thus,

$$\mathbf{x}_1 = e^{-0.5t}[cos(t) \ -sin(t)]^T + ie^{-0.5t}[sin(t) \ cos(t)]^T$$

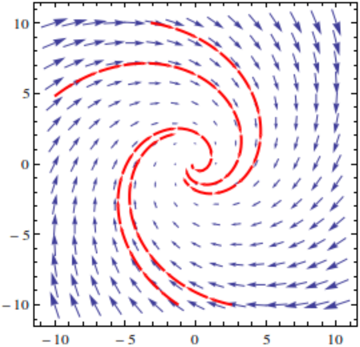
That is to say, the general solution can be found as

$$\mathbf{x} = c_1 e^{-0.5t}[cos(t) \ -sin(t)]^T + c_2 e^{-0.5t}[sin(t) \ cos(t)]^T$$

All trajectories approach the origin in this case. The trajectories will make infinitely many circuits about the origin, due to the sine and cosine factors.

Spiral Sink

For this case, where eigenvalues are complex, and of negative real parts, the origin is called a spiral sink. A spiral sink is always asymptotically stable.



Center

Let A be the following matrix

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues for this system are

$$\lambda_1 = i \ \lambda_2 = -i$$

The eigenvector corresponding to λ_1 is

$$\mathbf{v}_1 = [1 + i \ 1]^T$$

Thus one solution of this system is $\mathbf{x}_1 = c_1 \mathbf{v}_1 e^{it}$. To find real-valued solutions, we take the real and imaginary parts, respectively of $\mathbf{x}_1$. Thus,

$$\mathbf{x}_1 = [cos(t) - sin(t) \ cos(t)]^T + i[cos(t) + sin(t) \ sin(t)]^T$$

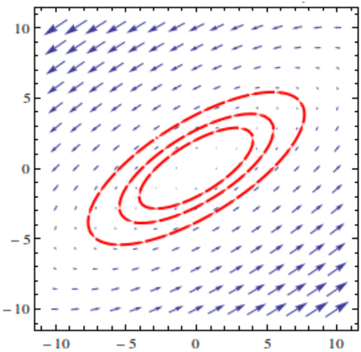
That is to say, the general solution can be found as

$$\mathbf{x} = c_1[cos(t) - sin(t) \ cos(t)]^T + c_2[cos(t) + sin(t) \ sin(t)]^T$$

All trajectories approach the origin in this case. The trajectories will make infinitely many circuits about the origin, due to the sine and cosine factors. Actually, these circuits are all concentric ellipses.

Center

For this case, where eigenvalues are complex, and of zero real parts, the origin is called a center. A center is always stable, but not asymptotically stable.



Summary for Stability

To conclude, we use the following table as a summary.

Eigenvalues	Critical Point	Stability
$\lambda_1 < 0 < \lambda_2$	saddle point	unstable
$\lambda_1 \geqslant \lambda_2 > 0^{(1)}$	nodal source	unstable
$\lambda_1 \leqslant \lambda_2 < 0^{(1)}$	nodal sink	asymptotically stable
$\lambda_1 = \lambda_2 > 0^{(2)}$	improper nodal source	unstable
$\lambda_1 = \lambda_2 < 0^{(2)}$	improper nodal sink	asymptotically stable
$\lambda_1, \lambda_2 = p \pm iq(p > 0)$	spiral source	unstable
$\lambda_1, \lambda_2 = p \pm iq(p = 0)$	center	stable
$\lambda_1, \lambda_2 = p \pm iq(p < 0)$	spiral sink	asymptotically stable

Note:

(1) with different eigenvectors

(2) with the same eigenvectors